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Weakly $*$-ordered $*$-fields

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Abstract

Let $(D, *)$ be a weakly $*$-ordered $*$-field and F' the field of symmetric elements in the center of D. We find a necessary and sufficient condition to extend an ordering of F' to a weak $*$-ordering of $(D, *)$. Using this, we prove that our notion of weak preorderings is the correct generalization of preorderings in ordered fields. We then study the notion of forms over weak preorderings and SAP preorderings. As a result, we can also determine when a semiordering of F' be extendable to a Baer ordering in $(D, *)$.

1. Introduction

This paper is a continuation of the two earlier papers [7, 8] on weakly $*$-ordered $*$-fields. By a $*$-field $(D, *)$, we shall mean a division ring D with an involution $*$ on D. Throughout this paper, D is always finite dimensional over its center F and the fixed field of $*$ in F is F'. In this case, it is proved in [7] that only standard quaternion algebras and $*$-fields which are odd dimensional over their centers admit weak $*$-orderings. Henceforth, we shall always assume either $(D, *)$ is a standard quaternion algebra or $[D:F]$ is odd. We denote the set of symmetric elements, nonzero symmetric elements, by $S(D, *)$ and $S(D^\times, *)$, respectively.

A Baer ordering P on $(D, *)$ is a subset of $S(D, *)$ satisfying $P + P \subset P$, $1 \in P$, $dPd^* \subset P$ for all $d \in D$, $P \cap (-P) = \{0\}$, an $P \cup (-P) = S(D, *)$. (Note that our definition here is slightly different from the usual definition of Baer orderings in [5, 3, 8]. To recover the usual definition, we just need to delete 0 from our definition of Baer orderings. We make this slight adjustment so that when $D = F$ and $* = id$, we get back the usual definition of semiorderings.) A Baer ordering P is called a weak $*$-ordering if $(P \cap F) \cdot P \subset P$. In [8], we define the notion of Baer preorderings which is analogous to the notion of pre-semicones defined in [9]. (Here, we assume pre-semicones contain 1.) In this paper, our first objective is to define the corresponding notion for weakly $*$-ordered $*$-fields, namely weak preorderings. It is not so difficult to guess the correct definition of weak preorderings. However, it is not possible to

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generalize the usual method to extend weak preorderings. A new approach has to be developed to deal with this problem. The key is to show that $\{xNrd(x): x \in S(D,*)\} \subset \tilde{\mathscr{J}}(\{1\})$ is the smallest Baer preordering in $(D,*)$. In order to prove that, we need to study when an ordering of F' can be extended to a weak $*$-ordering of $(D,*)$.

It is of independent interest to know when a semiordering of F' can be extended to $(D,*)$. Some general results toward this direction will be proved in Section 2. As it turns out, only certain normed semiorderings can be extended. That will be proved in Section 5. As we have mentioned earlier, our main concern is when $[D:F]$ is odd and the semiordering concerned is an ordering.

The first part of Section 3 is to prove that our definition of weak preordering is indeed the correct one. After that, we generalize some well-known results concerning preorderings in commutative fields. The key is to observe that for any weak preordering T on $(D,*)$, its restriction $T \cap F'$ is a preordering on F'. It turns out that the space of weak $*$-orderings that contain the weak preordering T is homeomorphic to the space of orderings that contain the preordering $T \cap F'$ in F'.

In Section 5, we first define the notion of T-forms. As expected, all corresponding results in the commutative case remain true. As a consequence, we show that the space of Baer orderings of $(D,*)$ is homeomorphic to the space of normed $T \cap F'$-semiorderings of F'. This allows us to generalize the characterization of SAP preorderings in $*$-fields, which is an improvement of [8, Theorem 5.6].

Let us first fix some notation used throughout the paper. A $*$-valuation v on $(D,*)$ is a valuation on D in the sense of [10] such that $v(a) = v(a^*)$ for all $a \in D^\times$. As usual, we denote the valuation ring, its maximal ideal, its group of units, residue class division ring and value group by R_v, M_v, U_v, $\bar{D}_v$ and Γ_D. For any element a in R_v, we shall denote its image in $\bar{D}_v$ by $\bar{a}$. If E is any subset of D, we denote $\{\bar{a}: a \in E \cap R_v\}$ by $\bar{E}$ and $v(E\backslash\{0\})$ by Γ_E. Note that $*$ induces an involution on $\bar{D}_v$ as follows: for $y = \bar{x}$ in $\bar{D}_v$, $\bar{*}(y) = \overline{x^*}$.

2. Extending Baer orderings from $(F,*)$ to $(D,*)$

Throughout this section, we shall assume $[D:F]$ is odd. Our main goal is to find a value theoretical condition under which a Baer ordering of $(F,*)$ is extendable to a Baer ordering of $(D,*)$. First, we prove a result on total valuation rings.

Lemma 2.1. *Let B be a total valuation ring of a division ring D and F the center of D. Let $J(B)$ be the maximal ideal of B and $V = F \cap B$. Denote the division rings $B/J(B)$ and $V/(F \cap J(B))$ by $\bar{B}$ and $\bar{V}$, respectively. Let $\mathscr{B} = \{B': B'$ is a total valuation ring in D with $B' \cap F = V\}$. Then $|\mathscr{B}| \cdot [\bar{B}:\bar{V}]$ is a divisor of $[D:F]$.*

Proof. By [12, Theorem C, G], we see that $n_B[\bar{B}:\bar{V}]$ is a divisor of $[D:F]$, where n_B is the number of conjugates of B in D. On the other hand, we conclude from

[1, Theorem 2] that if $B' \in \mathscr{B}$, then B' is a conjugate of B. Therefore, $|\mathscr{B}| = n_B$ and our lemma follows. $\square$

Recall that for any Baer ordering P of $(D, *)$,

$$A_P := \{\alpha \in D\colon n - \alpha\alpha^* \in P \text{ for some } n \in \mathbb{Z}\}$$

is a $*$-closed total valuation ring. We call A_P the order valuation ring of P. Let $J(A_P)$ be the maximal ideal of A_P. P induces an archimedean ordering $\bar{P}$ on the residue $*$-field $(A_P/J(A_P), \bar{*})$. Hence, $A_P/J(A_P)$ is isomorphic to a subfield of $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ [5]. By Lemma 2.1, we see that $[A_P/J(A_P) : (A_P \cap F)/J(A_P)]$ must be odd. This rules out the possibility that $A_P/J(A_P)$ is a quaternion algebra. We therefore obtain the following lemma.

Lemma 2.2. *Suppose $[D:F]$ is odd. Then $A_P/J(A_P)$ is a commutative $*$-ordered $*$-field.*

Let Q be a Baer ordering of $(F, *)$ and $A_Q = \{x \in F\colon n - xx^* \in Q \text{ for some } n \in \mathbb{Z}\}$. A_Q is a $*$-closed valuation ring in F. Let

$$\mathscr{B} = \{B\colon B \text{ is a total valuation ring of } D \text{ and } B \cap F = A_Q\}.$$

Suppose Q can be extended to a Baer ordering P of $(D, *)$. Then clearly, $A_P \in \mathscr{B}$. By Lemma 2.2, $A_P/J(A_P)$ is a commutative field. Since all elements in $\mathscr{B}$ are conjugates, it follows that $B/J(B)$ is a commutative field for all $B \in \mathscr{B}$. Naturally, we may ask if this condition is sufficient for the extension of Q to $(D, *)$. As we shall see, this turns out to be so if we further assume A_Q is compatible with Q.

Lemma 2.3. *Let $(S, *)$ be a $*$-field and E a $*$-closed subfield in the center of S. Suppose $v: S \to \Gamma_S$ is a $*$-valuation with $|\Gamma_S : \Gamma_E|$ being odd. If every Baer ordering of $(\bar{E}_v, \bar{*})$ can be extended to a Baer ordering of $(\bar{S}_v, \bar{*})$, then any Baer ordering compatible with $v|_E$ on $(E, *)$ can be extended to a Baer ordering compatible with v on $(S, *)$.*

Proof. Let Q be a Baer ordering compatible with $v|_E$ on $(E, *)$. Since $|\Gamma_S : \Gamma_E|$ is odd, we can assume the subset A required in [7, Proposition 3.2] is a subset of E. By the same proposition, Q induces a mapping $\sigma_Q : A \to \{\pm 1\}$ and a Baer ordering $\bar{Q}_d$ on $(\bar{E}_v, \bar{*})$ for every $d \in A$. By assumption, every $\bar{Q}_d$ can be extended to a Baer ordering $\bar{P}_d$ of $(\bar{S}_v, \bar{*})$. Using σ_Q and $\bar{P}_d$'s, we apply [7, Proposition 3.3] to get a desired Baer ordering of $(S, *)$. $\square$

Remark. It is clear that in the proof, we only need to assume $\bar{Q}_d$ is extendable for all $d \in A$. This fact will be needed in the proof of Theorem 2.5.

Corollary 2.4. *Let $(S, *)$ be a commutative $*$-field and $(E, *)$ a subfield of $(S, *)$. Suppose S is a Galois extension of E and $[S:E]$ is odd. Then for any $*$-valuation v of $(S, *)$ and*

Baer ordering Q compatible with $v|_E$ on $(E,*)$, Q can be extended to a Baer ordering compatible with v on $(S,*)$.

Proof. Since S is a Galois extension over E, $|\Gamma_S:\Gamma_E|$ and $[\bar{S}:\bar{E}]$ are both odd. It is obvious that every Baer ordering of $(\bar{E},\bar{*})$ can be extended to a Baer ordering of $(\bar{S},\bar{*})$. We then apply Lemma 2.3 to get a desired extension. $\square$

We now come back to the question we raised earlier. Let Q be a Baer ordering of $(F,*)$ and V a $*$-closed valuation ring of F that is compatible with the ordering Q. Let

$$\mathscr{B} = \{B\colon B \text{ is a total valuation ring of } D \text{ and } B\cap F = V\}.$$

By Lemma 2.1, $|\mathscr{B}|$ is odd. On the other hand, since V is $*$-closed, $\{B^*\colon B\in\mathscr{B}\} = \mathscr{B}$. By pairing up every B with B^* in $\mathscr{B}$, we conclude that a $*$-closed B exists in $\mathscr{B}$. (In fact, B is unique. Since we do not need this fact, we skip its proof.)

Theorem 2.5. *Suppose $[D:F]$ is odd and Q is a Baer ordering of $(F,*)$. Let V be a $*$-closed valuation ring of F and B an extension of V to D. If Q is compatible with V and $B/J(B)$ is a commutative field, then Q can be extended to a Baer ordering on $(D,*)$.*

Proof. We shall proceed by induction on $[D:F]$. It is trivial when $D = F$. As argued above, we may assume B is $*$-closed. By [1, Lemma 4] and [7, Lemma 2.2], there is a smallest $*$-closed invariant valuation ring R such that $D \supsetneq R \supset B$. Therefore, R induces a $*$-valuation v on $(D,*)$ with $R_v = R$. Note that $\bar{B} := \{a + M_v\colon a\in B\}$ is a $*$-closed total valuation ring with $\bar{B}\cap\bar{F} = \bar{V}$. Furthermore, it is obvious that $\bar{B}/J(\bar{B}) \cong B/J(B)$. Hence, $\bar{B}/J(\bar{B})$ is also a commutative field.

By [8, Lemma 1.4], $|\Gamma_D:\Gamma_{F'}|$ is odd. Thus, A defined in Lemma 2.3 can be assumed in F'. By the definition of R_v, $(R_v\cap F) \supset V$. Therefore, $v|_F$ is also compatible with Q. Hence, Q induces a Baer ordering $\bar{Q}_d$ of $(\bar{F}_v,\bar{*})$ for every $d\in A$. By the remark following Lemma 2.3, it suffices to show that every $\bar{Q}_d$ can be extended to $(\bar{D}_v,\bar{*})$.

Fix an element d in A. As V is compatible with Q, $\bar{V}$ is also compatible with $\bar{Q}_d$. Observe that $Z(\bar{D}_v)\cap\bar{B}$ is a $*$-closed extension of $\bar{V}$ to $Z(\bar{D}_v)$. Since $Z(\bar{D}_v)$ is a Galois extension of $\bar{F}_v$, we can apply Corollary 2.4 to obtain an extension $\bar{Q}'_d$ of $\bar{Q}_d$ such that $\bar{Q}'_d$ is compatible with $Z(\bar{D}_v)\cap\bar{B}$. If B is invariant, then we are done as $\bar{D}_v = B/J(B) = Z(\bar{D}_v)$. Otherwise, we conclude from [1, Lemma 5] that $[D:F] > [\bar{D}_v:Z(\bar{D}_v)]$. Since $\bar{B}/J(\bar{B})$ is a field, we can apply induction to obtain an extension of $\bar{Q}'_d$ to $(\bar{D}_v,\bar{*})$. $\square$

The above result has one drawback. Given Q and V, it is often not easy to determine B and $B/J(B)$. However, if it is given that $V = A_{P'}\cap F$ for some Baer ordering P' of $(D,*)$, then we see from Lemma 2.2 that $A_{P'}/J(A_{P'})$ is a commutative field. We can then conclude from Theorem 2.5 that Q is extendable to $(D,*)$. However, the extension of Q need not be P' as the assumption $A_Q = A_{P'}\cap F$ does not even imply $Q\subset P'$. We sum up the above as follows.

Corollary 2.6. *Let Q be a Baer ordering of $(F,*)$ with Q compatible with $A_{P'} \cap F$ for some Baer ordering P' of $(D,*)$. If $[D:F]$ is odd, then Q can be extended to a Baer ordering P of $(D,*)$.*

In later applications, we often consider extensions of orderings from F' to $(D,*)$. For convenience, we record the following corollary.

Corollary 2.7. *Suppose $[D:F]$ is odd and Q is an ordering of F'. Let $A_Q = \{a \in F': n - a^2 \in Q \text{ for some } n \in \mathbb{Z}\}$. If Q is also a $*$-ordering of $(F,*)$ and $A_Q = A_{P'} \cap F'$ for some Baer ordering P' of $(D,*)$, then there exists a unique Baer ordering P that extends Q. Moreover, P is a weak $*$-ordering on $(D,*)$.*

Proof. Since Q is compatible with A_Q, Q can be regarded as a $*$-ordering of $(F,*)$ compatible with $A_{P'} \cap F$. We can thus apply Corollary 2.6 to get an extension P of Q. By [8, Theorem 3.1], P is uniquely determined and P is a weak $*$-ordering. $\square$

3. Weak preorderings

For the convenience of readers, we recall some terminologies used in [8]. A function $J \in End_{F'}(S(D,*))$ is called positive if there exist $\alpha_1, \ldots, \alpha_n \in D^\times$ such that

$$J(x) = \alpha_1 x \alpha_1^* + \alpha_2 x \alpha_2^* + \cdots + \alpha_n x \alpha_n^* \quad \forall x \in S(D,*).$$

We denote the set of all positive functions on $(D,*)$ by $\mathscr{J}$.

In [8], it is shown that a $*$-field $(D,*)$ admits a Baer ordering iff there exists a subgroup G in $Aut_{F'}\, S(D,*)$ such that G is closed under addition and $G \supset \mathscr{J}$. If such a G exists, we then say $(D,*)$ is formally real and the smallest possible G is denoted by $\tilde{\mathscr{J}}$. In order to be consistent with the definition that Baer orderings contain 0, we modify the definition of Baer preordering as follows.

Definition 3.1. Let $(D,*)$ be a Baer formally real $*$-field. A *Baer preordering* T is a subset of $S(D,*)$ such that (i) $T + T \subset T$, (ii) $f(T) = T$ for all $f \in \tilde{\mathscr{J}}$, (iii) $T \cap (-T) = \{0\}$ and (iv) $1 \in T$.

Remark. T is a Baer preordering iff $T \backslash \{0\}$ is a Baer preordering in the sense of [8, Definition 2.4]. It is clear that all results concerned Baer preorderings in [8] remain true for our new notion of Baer preorderings.

For any subset A of $S(D,*)$, we denote $\{\sum f_i(a_i): f_i \in \tilde{\mathscr{J}}, a_i \in A\} \cup \{0\}$ by $\tilde{\mathscr{J}}(A)$. It is easy to see that $\tilde{\mathscr{J}}(A)$ is a Baer preordering iff for any nonzero a_i's in A, f_i's in $\tilde{\mathscr{J}}$, $\sum f_i(a_i) \neq 0$. So when $(D,*)$ is formally real, $\tilde{\mathscr{J}}(\{1\})$ is the smallest Baer preordering of $(D,*)$.

It has been proved in [7] that when a $*$-field $(D, *)$ admits a weak $*$-ordering, either $(D, *)$ is a standard quaternion algebra or $[D:F]$ is odd. There is a fundamental difference for weak $*$-orderings in these two cases. When $(D, *)$ is odd, [8, Corollary 3.4] states that for all $x \in S(D, *)$ and weak $*$-ordering P of $(D, *)$, $xNrd(x) \in P$. However, if $(D, *)$ is a standard quaternion algebra, it is no longer true that a weak $*$-ordering contains $\{xNrd(x)\colon x \in S(D, *)\}$. (In fact, since $deg\, D = 2$ in that case, a weak $*$-ordering $P \supset \{xNrd(x)\colon x \in S(D, *)\}$ implies $P = F'$. This is clearly impossible.) For this reason, many arguments that work for one case might not work for the other case.

Definition 3.2. Suppose $(D, *)$ is trivial or $[D:F]$ is odd. A Baer preordering T is called a *weak preordering* if $(T \cap F') \cdot T \subset T$.

When $(D, *)$ is trivial, weak preorderings are no more than ordinary preorderings in F'. In particular, all usual results that concern preorderings can be generalized. However, in case $[D:F]$ is odd, it is not clear that the usual argument of extending preorderings works for weak preorderings. Moreover, as we point out earlier, every weak $*$-ordering of $(D, *)$ contains the set $\{xNrd(x)\colon x \in S(D, *)\}$. Since we expect a weak preordering equals to the intersection of all weak $*$-orderings that contain it, a weak preordering should also contain the set $\{xNrd(x)\colon x \in S(D, *)\}$. Apparently, this does not follow easily from our definition of weak preorderings. In order to understand the situation, we consider $\tilde{\mathscr{J}}(\{1\})$, the smallest Baer preordering of $(D, *)$. On one hand, $\tilde{\mathscr{J}}(\{1\})$ is a weak preordering. (For any nonzero element a, b in $\tilde{\mathscr{J}}(\{1\})$, $a = f(1)$ and $b = g(1)$ for some $f, g \in \tilde{\mathscr{J}}$. Therefore, if $a \in F'$ also, $ab = ag(1) = g(a) = g(f(1)) \in \tilde{\mathscr{J}}(\{1\})$.) On the other hand, T_0, the intersection of all weak $*$-orderings of $(D, *)$, should be the smallest weak preordering of $(D, *)$. This suggests that $\tilde{\mathscr{J}}(\{1\}) = T_0$. Once we prove this, we can then conclude all Baer preorderings contain the set $\{xNrd(x)\colon x \in S(D, *)\}$. So the key is to prove that $\tilde{\mathscr{J}}(\{1\}) = T_0$.

Lemma 3.3. $T_0 \cap F' = \tilde{\mathscr{J}}(\{1\}) \cap F'$.

Proof. Observe that $\tilde{\mathscr{J}}(\{1\}) \cap F'$ is a preordering of F'. If $(D, *)$ is trivial, any ordering of F' that extends $\tilde{\mathscr{J}}(\{1\})$ is in fact a weak $*$-ordering of $(D, *)$. It follows that $T_0 = \tilde{\mathscr{J}}(\{1\})$. Next, we assume $[D:F]$ is odd. Let $a \in F' \backslash \tilde{\mathscr{J}}(\{1\})$. Consider the set $T := \tilde{\mathscr{J}}(\{1\}) - a\tilde{\mathscr{J}}(\{1\})$. Clearly, T satisfies (i), (ii) and (iv) of Definition 3.1. To prove (iii) is also satisfied, it suffices to show that if there exist $x, y \in \tilde{\mathscr{J}}(\{1\})$ such that $x - ya = 0$, then $x = y = 0$. Note that $x = f(1)$ and $y = g(1)$ for some $f, g \in \tilde{\mathscr{J}}$. Using the equation $x - ya = 0$, we obtain $f(1) = ag(1) = g(a)$. As $\tilde{\mathscr{J}}$ is a subgroup of $Aut_{F'}\, S(D, *)$, we get $a = g^{-1}f(1) \in \tilde{\mathscr{J}}(\{1\})$. This contradicts our assumption on a. So, T is a Baer preordering. By [8, Corollary 2.7], T can be extended to a Baer ordering P on $(D, *)$. Let $T' = \tilde{\mathscr{J}}(\{1\}) \cap F' - a(\tilde{\mathscr{J}}(\{1\}) \cap F')$. As $\tilde{\mathscr{J}}(\{1\}) \cap F'$ is a preordering of F', T' is also a preordering of F'. Clearly, $P \supset T'$. In particular, $(1 + J(A_P)) \cap -T' = \emptyset$. Therefore, T' is compatible with the valuation ring $A_P \cap F'$. Hence, there exists an ordering $Q \supset T'$ such that Q is compatible with $A_P \cap F'$ on F'. Note that Q is also a $*$-ordering of $(F, *)$ as $Q \supset \tilde{\mathscr{J}}(\{1\}) \cap F'$. Therefore, by Corollary 2.7, Q can be extended to a weak $*$-ordering P_a on $(D, *)$. Clearly, $-a \in Q \subset P_a$. Therefore $a \notin T_0 \cap F'$. □

Before we continue with the proof $\tilde{\mathscr{J}}(\{1\}) = T_0$, let us deduce an important result from the above lemma. Let X_D^* be the space of all weak $*$-orderings of $(D, *)$. Recall that $\tilde{\mathscr{J}}(\{1\}) \cap F'$ is a preordering of the field F'. We denote the space of all orderings of F' that contain $\tilde{\mathscr{J}}(\{1\}) \cap F'$ by $X/(\tilde{\mathscr{J}}(\{1\}) \cap F')$. Define a mapping $\phi: X_D^* \to X/(\tilde{\mathscr{J}}(\{1\}) \cap F')$ such that $\phi(P) = P \cap F'$ for all $P \in X_D^*$. We define the Harrison topologies on X_D^* and $X/(\tilde{\mathscr{J}}(\{1\}) \cap F')$ by using $\{H(x): x \in S(D^\times, *)\}$ and $\{H'(x): x \neq 0$, and $x \in F'\}$ as subbases for X_D^* and $X/(\tilde{\mathscr{J}}(\{1\}) \cap F')$, respectively. (Here, $H(x) = \{P \in X_D^*: x \in P\}$ and $H'(x) = \{Q \in X/(\tilde{\mathscr{J}}(\{1\}) \cap F'): x \in Q\}$.) With respect to the Harrison topologies in their respective spaces, the mapping ϕ is clearly continuous.

Theorem 3.4. *The mapping $\phi: X_D^* \to X/(\tilde{\mathscr{J}}(\{1\}) \cap F')$ defined above is a homeomorphism. In particular, an ordering Q of F' is extendable to a weak $*$-ordering of $(D, *)$ iff $Q \supset (\tilde{\mathscr{J}}(\{1\}) \cap F')$.*

Proof. When $(D, *)$ is trivial, the statement is trivial. In fact, for any $P \in X/(\tilde{\mathscr{J}}(\{1\}) \cap F')$, P can be regarded as a weak $*$-ordering of $(D, *)$. Hence, we may assume $[D:F]$ is odd. Applying [8, Theorem 3.1], we see that ϕ is injective. Next, we prove that ϕ is surjective. Observe that X_D^* is compact and ϕ is continous. Therefore, $\phi(X_D^*)$ is also compact and hence closed in $X/(\tilde{\mathscr{J}}(\{1\}) \cap F')$.

For the rest of the proof, we shall follow the notation of [6, Chapter 9]. Let $\lambda: X_{F'} \to M_{F'}$ be the map that sends an ordering $Q \in X_{F'}$ to $\lambda(Q)$, its associated real-valued place. Let $Q \in X/(\tilde{\mathscr{J}}(\{1\}) \cap F') \backslash \phi(X_D^*)$. We claim that $\lambda(Q) \notin \lambda(\phi(X_D^*))$. Otherwise, there exists $P \in X_D^*$ such that $\lambda(Q) = \lambda(P \cap F')$. In particular, Q is compatible with $A_P \cap F'$. Since $Q \supset \tilde{\mathscr{J}}(\{1\}) \cap F'$, we may regard Q as a $*$-ordering of $(F, *)$. Thus, by Corollary 2.7, Q can be extended to a weak $*$-ordering of $(D, *)$. This contradicts our assumption that $Q \notin \phi(X_D^*)$ and our claim is proved.

As $\lambda(Q) \notin \lambda(\phi(X_D^*))$, we apply [6, Proposition 9.13] to $\{Q\}$ and $\phi(X_D^*)$ to get an element $a \in F'$ that separates $\{Q\}$ from $\phi(X_D^*)$. We may also assume $-a \in Q$ and $a \in Q'$ for all $Q' \in \phi(X_D^*)$. By Lemma 3.3, $a \in \tilde{\mathscr{J}}(\{1\}) \cap F'$. Therefore, $\tilde{\mathscr{J}}(\{1\}) \cap F' \not\subseteq Q$. This is impossible. So we complete the proof that ϕ is surjective. Finally, using $H(x) = H(Nrd(x))$ [8, Corollary 3.4] and the surjectivity of ϕ, we conclude that $\phi(H(x)) = H'(Nrd(x))$ for all $x \in S(D^\times, *)$. This proves ϕ is an open mapping. Hence ϕ is a homeomorphism. □

Next, we improve Lemma 3.3.

Proposition 3.5. *Let T_0 be as defined before; $T_0 = \tilde{\mathscr{J}}(\{1\})$. In particular, $\tilde{\mathscr{J}}(\{1\}) \supset \{xNrd(x): x \in S(D, *)\}$ when $[D:F]$ is odd.*

Proof. The result is just Lemma 3.3 when $(D,*)$ is trivial. So we may assume $[D:F]$ is odd. It suffices to show that $T_0 \subset \tilde{\mathscr{J}}(\{1\})$. Let $a \in T_0$. Consider the subfield $F'(a)$ and the set $T' := \{\sum a_i x_i^2 \colon a_i \in \tilde{\mathscr{J}}(\{1\}) \cap F',\ x_i \in F'(a)\}$. As observed before, $\tilde{\mathscr{J}}(\{1\}) \cap F'$ is a preordering in F'. So T' is also a preordering in $F'(a)$. Clearly, $T' \subset \tilde{\mathscr{J}}(\{1\})$. If $a \notin T'$, then there exists an ordering Q of $F'(a)$ that contains $T'[-a]$. By Theorem 3.4, $Q \cap F'$ can be extended to a weak $*$-ordering P of $(D,*)$. But by [8, Theorem 3.1], $P \cap F'(a)$ is the unique extension of $Q \cap F'$. Consequently, $Q = P \cap F'(a)$. Therefore, $-a \in P$. This is impossible as by assumption $a \in T_0 \subset P$. This shows a $\in \tilde{\mathscr{J}}(\{1\})$. The last statement is now obvious as $T_0 \supset \{xNrd(x)\colon x \in S(D,*)\}$ when $[D:F]$ is odd. □

As a consequence of Proposition 3.5, we see that every Baer preordering contains the set $\{xNrd(x)\colon x \in S(D,*)\}$ when $[D:F]$ is odd. Moreover, $xNrd(x)^{-1} = xNrd(x)Nrd(x)^{-2} \in T_0 = \tilde{\mathscr{J}}(\{1\})$. Hence, there exists $f \in \tilde{\mathscr{J}}$ such that $xNrd(x)^{-1} = f(1)$. In particular, $x = f(Nrd(x))$. Since $f(T) = T$ for any Baer preordering T of $(D,*)$, we conclude that $x \in T$ iff $Nrd(x) \in T$. Therefore, $T = \{x \in S(D,*)\colon Nrd(x) \in T \cap F'\}$. Now let $a \in \tilde{\mathscr{J}}(\{1\}) \cap F'$ and $b \in T \cap F'$. Since $a = f(1)$ for some $f \in \tilde{\mathscr{J}}$, $ab = f(1)b = f(b) \in T$. This shows that $T \cap F'$ is a $\tilde{\mathscr{J}}(\{1\}) \cap F'$-module [6, Chapter 14]. By [6, Proposition 14.1], $\tilde{\mathscr{J}}(\{1\}) \cap F'$ is anisotropic. We sum up the above as follows.

Corollary 3.6. *Suppose $[D:F]$ is odd and T is a Baer preordering of $(D,*)$. Then $T \supset \{xNrd(x)\colon x \in S(D,*)\}$ and $T = \{x \in S(D,*)\colon Nrd(x) \in T \cap F'\}$. Furthermore, $T \cap F'$ is an anisotropic $\tilde{\mathscr{J}}(\{1\}) \cap F'$-module.*

The converse of the above corollary is also true. This is quite surprising as it implies that any normed $\tilde{\mathscr{J}}(\{1\}) \cap F'$-semiordering of F' can be extended to $(D,*)$. We shall defer the proof to Section 5, after we have introduced the notion of T-forms. In the mean time, we proceed to prove some basic results concerning weak preorderings.

Proposition 3.7. *Let T be a Baer preordering T of $(D,*)$. The following statements are equivalent:*
 (i) *T is a weak preordering.*
 (ii) *$T \cap F'$ is a preordering of F'.*
 (iii) *For any $x, y \in T$, $xy = yx$ implies $xy \in T$.*

Proof. (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (i) are obvious. To prove (ii) $\Rightarrow$ (iii), we can assume $(D,*)$ is nontrivial, i.e. $[D:F]$ is odd. Observe that if $x, y \in T$ with $xy = yx$, then $xy \in S(D,*)$ and $Nrd(xy) = Nrd(x)Nrd(y) \in T \cap F'$. So by Corollary 3.6, $xy \in T$. □

Corollary 3.8. *Let T and S be Baer preorderings of $(D,*)$. Then $T \subset S$ iff $T \cap F' \subset S \cap F'$. In particular, T is a Baer ordering (resp. weak $*$-ordering) of $(D,*)$ iff $T \cap F'$ is a semiordering (resp. an ordering) in F'. Thus, any Baer ordering P of $(D,*)$ is uniquely determined by $P \cap F'$.*

Proof. The result is clear when $(D,*)$ is trivial. We only need to deal with the case when $[D:F]$ is odd. The first assertion clearly follows from the fact that $T = \{x \in S(D,*)\colon Nrd(x) \in T \cap F'\}$ and $S = \{x \in S(D,*)\colon Nrd(x) \in S \cap F'\}$. If $T \cap F'$ is a semiordering, then for any $x \in S(D,*)$, $Nrd(x) \in (T \cap F') \cup -(T \cap F')$. Therefore, $x \in T \cup (-T)$. This shows that T is a Baer ordering. Lastly, by Proposition 3.7, we see that T is a weak $*$-ordering when $T \cap F'$ is an ordering of F'. □

Let T be a weak preordering of $(D,*)$ and X_T the space of weak $*$-orderings that contain T. $T \cap F'$ is a preordering of the field F'. We define $X/(T \cap F')$ to be the space of orderings on F' that contain $T \cap F'$. Let ϕ be the mapping as defined in Theorem 3.4 and ϕ_T the restriction of ϕ on X_T. Obviously, ϕ_T is continuous and its range is in $X/(T \cap F')$. Conversely, for any $Q \in X/(T \cap F')$, $\phi^{-1}(Q) \cap F' = Q$. When $(D,*)$ is trivial, $\phi^{-1}(Q) = Q \supset T$, whereas in case $[D:F]$ is odd, $\phi^{-1}(Q) \supset T$ follows from Corollary 3.8.

Proposition 3.9. *Let T be a weak preordering of $(D,*)$. $\phi_T\colon X_T \to X/(T \cap F')$ defined above is a homeomorphism. Moreover, $T = \bigcap_{P \in X_T} P$.*

Proof. We only need to prove the last statement. Since ϕ_T is surjective, it is clear that $(\bigcap_{P \in X_T} P) \cap F' = T \cap F'$. As both $(\bigcap_{P \in X_T} P)$ and T are weak preorderings, Corollary 3.8 implies that they must be equal. □

Using a similar argument as above, we show that for a preordering T' of F' that contains $\tilde{\mathscr{J}}(\{1\}) \cap F'$, $T := \bigcap_{P \in \phi^{-1}(X/T')} P$ is a weak preordering and $P \cap F' = T'$. Therefore, $T := \{x \in S(D,*)\colon Nrd(x) \in T'\}$. This proves a partial converse of Corollary 3.6. We record the above as follows.

Lemma 3.10. *Let T' be a preordering of F' and $T' \supset \tilde{\mathscr{J}}(\{1\}) \cap F'$. Then $\{x \in S(D,*)\colon Nrd(x) \in T'\}$ is a weak preordering of $(D,*)$.*

To end this section, we generalize the notion of fans to $*$-fields.

Definition 3.11. A weak preordering T is called a *fan* if for any $S \subset S(D,*)$ that contains T, S is in fact a weak $*$-ordering whenever the following conditions are satisfied:

(i) $S \cup (-S) = S(D,*)$,

(ii) $S \cap -S = \{0\}$,

(iii) for all $x, y \in S$ with $xy = yx$, we have $xy \in S$.

Proposition 3.12. *A weak preordering T is a fan of $(D,*)$ iff $T \cap F'$ is a fan in F'.*

Proof. If $(D,*)$ is trivial, the result is obvious. So we may assume $[D:F]$ is odd. Suppose T is a fan. Let S' be a subset of F' such that $S' \supset T \cap F'$, $-1 \notin S'$ and $S' \backslash \{0\}$ is a subgroup of index 2 in $F'\backslash\{0\}$. Let $S := \{x \in S(D,*): Nrd(x) \in S'\}$. Obviously, $S \supset T$ and S satisfies the conditions listed in Definition 3.11. Since T is a fan, S is a weak $*$-ordering. As $deg\, D$ is odd, $S \cap F' = S'$. Therefore, $S' = S \cap F'$ is an ordering. This proves that $T \cap F'$ is a fan.

Conversely, suppose $T \cap F'$ is a fan and S is a subset of $S(D,*)$ that contains T and satisfies the three conditions in Definition 3.11. Then in particular, $S \cap F' \supset T \cap F'$ and $(S \cap F')\backslash\{0\}$ is a subgroup of index 2 in $F'\backslash\{0\}$ with $-1 \notin S \cap F'$. Since $T \cap F'$ is a fan, $S \cap F'$ is an ordering containing $T \cap F'$. By Corollary 3.6 and Lemma 3.10, $P := \{x \in S(D,*): Nrd(x) \in S \cap F'\}$ is a weak $*$-ordering of $(D,*)$. It remains to show $P = S$. Using condition (i), it is enough to show $S \subset P$. For any $x \in S$, $x^{-1}Nrd(x) \in T \subset S$. Thus by (ii), we conclude $Nrd(x) = x(x^{-1}Nrd(x)) \in S \cap F'$. By the definition of P, $x \in P$. □

In [2], many known results on fans in commutative fields have been generalized. Indeed, most of the known results can be generalized to our notion of fans in $*$-fields. In many cases, proofs can be carried through from the commutative case without much changes.

4. Compatibility between $*$-valuations and preorderings

In the literature, the notion of compatibility is not standardized. So to avoid unnecessary confusion, we shall recall some definitions in [7]. A $*$-valuation v on $(D,*)$ is said to be compatible with a Baer ordering P if for all $a,b \in P$, $a - b \in P$ implies $v(b) \geq v(a)$. However, it is known that even in case when D is a field, the valuation induced by the order valuation ring A_P is not necessary compatible with P. To deal with this kind of situation, we introduce in [7] the notion of semicompatibility. We say a $*$-valuation v is semicompatible with a Baer ordering P if the valuation ring R_v contains A_P. It is easy to see that if v is compatible with P, then v is also semicompatible with P. Moreover, the converse holds when P is a weak $*$-ordering [7, Theorem 4.11]. In this section, we shall extend the notion of compatibility to preorderings.

Definition 4.1. A $*$-valuation v is said to be *semicompatible* (resp. *compatible*) with a Baer (resp. weak) preordering T if v semicompatible with a Baer ordering (resp. weak $*$-ordering) P that contains T in $(D,*)$.

Proposition 4.2. *Suppose v is a $*$-valuation semicompatible (resp. compatible) with a Baer (resp. weak) preordering T. Then $\bar{T} = \{\bar{x}: x \in T \cap R_v\}$ is a Baer (resp. weak) preordering.*

Proof. By assumption there exists a Baer ordering P of $(D,*)$ such that P is semicompatible with v. Hence, by [7, Corollary 2.6], $\bar{P} = \{\bar{x}: x \in P \cap R_v\}$ is a Baer ordering of $(\bar{D}_v, \bar{*})$. In particular, $(\bar{D}_v, \bar{*})$ is Baer formally real. Clearly, $\bar{T}$ satisfies (i), (iii) and (iv) of Definition 3.1.

Let $\mathscr{C} := \{f \in \tilde{\mathscr{J}}: f(P \cap R_v) = P \cap R_v \text{ and } f(P \cap M_v) = P \cap M_v\}$. Obviously, $\mathscr{C}$ is a subgroup of $\tilde{\mathscr{J}}$. We claim that $\mathscr{C}$ is also closed under addition. It suffices to show that for any $a \in P \cap U_v$ and $f, g \in \mathscr{C}$, $(f+g)(a) \in P \cap U_v$. Note that by the definition of C, $f(a)$ and $g(a)$ are in $P \cap U_v$. Hence $(f+g)(a) > f(a)$. It follows from [7, Proposition 2.5] that $(f+g)(a) \in P \cap U_v$. For any $f \in \mathscr{C}$, we define $\bar{f}: S(\bar{D}_v, \bar{*}) \to S(\bar{D}_v, \bar{*})$ such that $\bar{f}(\bar{x}) = \overline{f(x)}$ for all $x \in R_v \cap S(D,*)$. Clearly, $\bar{\mathscr{C}} := \{\bar{f}: f \in \mathscr{C}\}$ is a subgroup of $Aut_{\bar{F}'} S(\bar{D}_v, \bar{*})$ and $\bar{\mathscr{C}}$ contains all positive functions on $S(\bar{D}_v, \bar{*})$. On the other hand, as $\mathscr{C}$ is closed under addition, so is $\bar{\mathscr{C}}$. It follows that $\bar{\mathscr{C}}$ contains all totally positive functions on $S(\bar{D}_v, \bar{*})$. Consequently, for any $x \in U_v \cap T$ and totally positive function $\bar{g}$ on $S(\bar{D}_v, \bar{*})$, there exists $f \in \mathscr{C}$ such that $\bar{g}(\bar{x}) = \overline{f(x)}$. Since $f(x) \in T$, $\bar{g}(\bar{x}) \in \bar{T}$. This proves that $\bar{T}$ is a Baer preordering.

Suppose T is a weak preordering. $\bar{T}$ is already a Baer preordering. To show $\bar{T}$ is a weak preordering, it suffices to show that $Z(\bar{D}_v) \cap \bar{T}$ is a preordering. It is certainly the case when $(D,*)$ is trivial. Thus, we may assume $[D:F]$ is odd. Let $a, b \in T \cap U_v$ with $\bar{a}, \bar{b} \in Z(\bar{D}_v)$.

Consider the symmetric $ab + ba$. Since $Nrd((ab+ba)/2) = Nrd(a)Nrd(b)Nrd((1 + b^{-1}a^{-1}ba)/2) \in F'$, $Nrd(1 + b^{-1}a^{-1}ba) \in F'$. As $\bar{a}$ and $\bar{b}$ commutes, $(1 + b^{-1}a^{-1}ba)/2 = 1 + z$ for some $z \in M_v$. By an old lemma of Wedderburn [13, Lemma 4], we see that $N_{F(1+z)/F}(1+z) \in 1 + M_v$. Therefore, $Nrd(1+z) = 1 + y$ for some $y \in F' \cap M_v$. As $deg\, D$ is odd, $Nrd((1+y)(ab+ba)/2) = Nrd(a)Nrd(b)(1+y)^{2r}$ for some integer r. By Lemma 3.10, $(1+y)(ab+ba)/2 \in T$. Hence $\bar{a}\bar{b} \in \bar{T}$. □

Many results concerned with the compatibility between preorderings and valuations can be generalized. To illustrate this, we generalize two fundamental results in the commutative case.

Corollary 4.3. *Let T be a weak preordering and v a $*$-valuation of $(D,*)$. Then T is compatible with v iff $\bar{T}$ is a weak preordering iff $(1 + M_v) \cap -T = \emptyset$.*

Proof. By the above proposition, it suffices to show that $(1 + M_v) \cap -T = \emptyset$ implies v is compatible with T. Observe that $(1 + M_v) \cap -T = \emptyset$ implies $v|_{F'}$ is compatible with the preordering $T \cap F'$ of F'. By [6, Theorem 3.1], $v|_{F'}$ is compatible with an ordering Q of F' that contains $T \cap F'$. Using Proposition 3.9, we can extend Q to a weak $*$-ordering P of $(D,*)$ that contains T. It remains to prove that P is compatible with v. By [7, Theorem 4.11], it is enough to show that P is semicompatible with v. Note that we proved earlier $Nrd(1 + M_v) \subset (1 + M_v) \cap F'$. As $v|_{F'}$ is compatible with Q, $Nrd(1 + M_v) \subset Q$. Therefore, $(1 + M_v) \cap S(D^\times, *) \subset P$. By [7, Proposition 2.5], v is semicompatible with P. □

Proposition 4.4. *Let T be a weak preordering compatible with a $*$-valuation v on $(D,*)$. Suppose $\bar{S}$ is a weak preordering that contains $\bar{T}$. Let*

$$T \wedge \bar{S} := \{ts: t \in T,\ s \in F \cap U_v \text{ and } \bar{s} \in \bar{S}\}.$$

Then $T \wedge \bar{S}$ is also a weak preordering of $(D,*)$.

Proof. We may only concern with the case when $[D:F]$ is odd. Note that $(T \wedge \bar{S}) \cap F'$ is the wedge product of the preordering $T \cap F'$ and $\bar{S} \cap \bar{F}'$ in F'. Therefore, $(T \wedge \bar{S}) \cap F'$ is a preordering of F' that contains $T \cap F'$. By Lemma 3.10, $T' := \{x \in S(D, *): Nrd(x) \in (T \wedge \bar{S}) \cap F'\}$ is a weak preordering. Since T' contains T and $\{s \in F' \cap U_v: \bar{s} \in \bar{S}\}$, it follows from the definition of a weak preordering that $T' \supset T \wedge S'$. Conversely, if $x \in S(D^\times, *)$ and $Nrd(x) \in (T \wedge \bar{S}) \cap F'$, then $Nrd(x) = ts$, where $t \in T \cap F'$, $s \in F \cap U_v$ with $\bar{s} \in \bar{S}$. As $xNrd(x)t^{-1} \in T$, we obtain $x = xNrd(x)t^{-1}s^{-1} \in (T \wedge \bar{S})$. This proves $T' = T \wedge \bar{S}$. $\square$

To end this section, we sketch how the notion of compatibility be extended to total valuation rings.

Definition 4.5. A total valuation ring B is said to be *semicompatible* with a Baer ordering P if $B \supset A_P$.

Note that if B is semicompatible with a Baer ordering P, then B is $*$-closed. Otherwise, there exists $a \in B$ such that $a^* \notin B$. In particular, $a^*a^{-1} \notin A_P$. As A_P is a total valuation ring, $aa^{*-1} \in A_P$. Since A_P is $*$-closed, $a^{-1}a^* \in A_P \subset B$. It follows that $a^* \in B$. This is a contradiction.

In general, a total valuation ring B does not necessarily induce a valuation as defined in [10]. However, Gräter [4] proved that there is a chain of total valuation rings

$$D := R_0 \supset R_1 \supset R_2 \supset \cdots \supset R_i \cdots \supset R_t = B$$

such that for $t - 1 \geq j \geq 1$, $R_{j+1}/J(R_j)$ is the smallest invariant valuation ring in $R_j/J(R_j)$ that contains $B/J(R_j)$.

Suppose B is semicompatible with P. Set $P = P_0$. For a Baer ordering P_j semicompatible with $R_{j+1}/J(R_j)$ on $(R_j/J(R_j), \bar{*})$, P_j induces a Baer ordering P_{j+1} semicompatible with $R_{j+2}/J(R_j)$ on $(R_{j+1}/J(R_j), \bar{*})$. Starting from $t = 0$, we obtain inductively a sequence of Baer orderings P_j on $(R_j/J(R_j), \bar{*})$ for $j = 0, \ldots, t-1$. We define B to be compatible with P if each P_j is compatible with $R_{j+1}/J(R_j)$ for $j = 0, \ldots, t-1$. To generalize results regarding the compatibility between $*$-valuations and Baer orderings to total valuation rings, we simply apply induction on t defined above.

5. *T*-Forms and SAP preorderings

Let T be a fixed weak preordering. A T-form of dimension n is a formal expression $\varphi = \langle a_1, \ldots, a_n \rangle_T$ where a_i's are in $S(D^\times, *)$. Except for tensor product of T-forms, we can define the notion of orthogonal sum; P-signature; "signature function"; T-isometric and T-hyperbolic forms as in [3]. (Needless to say, weak $*$-orderings are used instead of $*$-orderings in all the definitions concerned.) As for tensor product of T-forms, we define $\langle a_1, \ldots, a_n \rangle_T \otimes \langle b_1, \ldots, b_n \rangle_T = \langle a_1 b_1, \ldots, a_i b_j, \ldots, a_n b_n \rangle_T$ if every a_i commutes with every b_j. In particular, it is defined when all a_i's are in F. To follow the usual notation, we say $\phi \cong_T \phi'$ if two T-forms ϕ and ϕ' are T-isometric.

Recall that in case $(D, *)$ is trivial, $S(D, *) = F'$ and T is just a preordering of F'. All the definitions above reduce to the ordinary definition of forms over a preordering in F'. Therefore, we can discard these cases and concentrate on the case when $[D:F]$ is odd. From now on, we shall assume $[D:F]$ is *odd*, even though some of the following results are also true when $[D:F] = 4$.

Lemma 5.1. (i) $\langle a_1, \ldots, a_n\rangle_T \cong_T \langle a_1 t_1, \ldots, a_n t_n\rangle_T$ *if* $t_i \in T$ *and* $a_i t_i = t_i a_i$ *for all* i. *In particular,* $\langle a_1, \ldots, a_n\rangle_T \cong_T \langle Nrd(a_1), \ldots, Nrd(a_n)\rangle_T$.

(ii) $\langle a, b\rangle_T \cong_T \langle a + b, Nrd(a)Nrd(b)(a + b)\rangle_T$ *if* $a + b \neq 0$.

Proof. (i) The first assertion is obvious. As for the second, it follows from the first and the fact that $x^{-1}Nrd(x) \in T$ for all $x \in S(D^\times, *)$. (Note that $[D:F]$ is odd).

Let P be a weak $*$-ordering containing T. If P contains both a, b, then $a + b \in P$ and $Nrd(a), Nrd(b) \in P \cap F'$. Hence $sgn_P\langle a, b\rangle_T = sgn_P\langle a + b, Nrd(a)Nrd(b)(a + b)\rangle_T = 2$. The case when P contains both $-a$, $-b$ can be verified similarly. If P contains a but not b, then $a + b, Nrd(a)Nrd(b)(a + b)$ must be of opposite sign. Therefore, $sgn_P\langle a, b\rangle_T = sgn_P\langle a + b, Nrd(a)Nrd(b)(a + b)\rangle_T = 0$. $\square$

Lemma 5.2. $\langle a_1, \ldots, a_n\rangle_T \cong_T \langle b_1, \ldots, b_n\rangle_T$ *iff*

$$\langle Nrd(a_1), \ldots, Nrd(a_n)\rangle_{T \cap F'} \cong_{T \cap F'} \langle Nrd(b_1), \ldots, Nrd(b_n)\rangle_{T \cap F'}.$$

Proof. By Lemma 5.1(i), we see that $\langle a_1, \ldots, a_n\rangle_T \cong_T \langle b_1, \ldots, b_n\rangle_T$ iff

$$\langle Nrd(a_1), \ldots, Nrd(a_n)\rangle_T \cong_T \langle Nrd(b_1), \ldots, Nrd(b_n)\rangle_T.$$

The last condition is equivalent to the condition

$$\langle Nrd(a_1), \ldots, Nrd(a_n)\rangle_{T \cap F'} \cong_{T \cap F'} \langle Nrd(b_1), \ldots, Nrd(b_n)\rangle_{T \cap F'},$$

because by Proposition 3.9, every ordering of $X_{T \cap F'}$ can be extended to a weak $*$-ordering in X_T. $\square$

Observe that two forms $\langle a\rangle_T$ and $\langle b\rangle_T$ are T-isometric iff $\langle Nrd(a)\rangle_{T \cap F'} \cong_{T \cap F'} \langle Nrd(b)\rangle_{T \cap F'}$. In particular, $Nrd(b) = Nrd(a)t$ for some $t \in T \cap F'$. Therefore, $b = a(a^{-1}Nrd(a)t)(Nrd(b)^{-1}b)$. Note that $a^{-1}Nrd(a)t$ and $Nrd(b)^{-1}b$ are both in T. This motivates us to define the following definition.

Definition 5.3. A T-form $\varphi = \langle a_1, \ldots, a_n\rangle_T$ is called *T-isotropic* if there exist elements t_i's and z_i's in T, such that not all $t_i z_i$'s are 0 and $\sum a_i t_i z_i = 0$. Otherwise, the T-form is called *T-anisotropic*. We say $d \in S(D^\times, *)$ is *T-represented* by φ if d lies in the set

$$D_T(\varphi) = \left\{\sum a_i t_i z_i \colon t_i, z_i \in T \text{ and } a_i t_i z_i \in S(D, *)\right\}.$$

Lemma 5.4. *A T-form $\varphi = \langle a_1, \ldots, a_n\rangle_T$ is isotropic iff there exist elements t_i's in T, not all equal to 0, such that $\sum Nrd(a_i)t_i = 0$. Moreover, $D_T(\varphi) = \{\sum Nrd(a_i)t_i\colon t_i \in T\}$.*

Proof. It suffices to show that

$$Nrd(a)T = \{at_1t_2\colon t_1, t_2 \in T \text{ and } at_1t_2 \in S(D,*)\}.$$

To prove '$\subset$', just use the fact that $a^{-1}Nrd(a) \in T$. Conversely, let t_1, t_2 be in T with $x = at_1t_2 \in S(D,*)$. Clearly, $Nrd(x) = Nrd(a)t$ for some $t \in T \cap F'$. Since $Nrd(a)^{-1}x$ is symmetric and its norm is in T, we conclude from Corollary 3.6 that $Nrd(a)^{-1}x \in T$. Therefore, $x \in Nrd(a)T$. □

Remark. It is more convenient to use Lemma 5.4 for the definitions of T-isotropic forms and the set of elements represented by a T-form. But then it will not be applicable when $(D,*)$ is a quaternion algebra.

Theorem 5.5 (Representation Criterion). *Suppose $b_1 \in S(D^\times,*)$ and $\phi = \langle a_1, \ldots, a_n\rangle_T$. Then $b_1 \in D_T(\phi)$ iff $\phi \cong_T \langle b_1, \ldots, b_n\rangle_T$ for suitable $b_2, \ldots, b_n \in S(D^\times,*)$. In particular, $D_T(\phi)$ depends only on the T-isometry class of ϕ.*

Proof. Suppose $b_1 \in D_T(\phi)$. Using the two types of T-isometries in Lemma 5.1 and a similar argument as in the proof of [6, 1.19], we obtain $\phi \cong_T \langle b_1, \ldots, b_n\rangle_T$. On the other hand, if $\phi \cong_T \langle b_1, \ldots, b_n\rangle_T$, then by Lemma 5.2, we have

$$\langle Nrd(a_1), \ldots, Nrd(a_n)\rangle_{T\cap F'} \cong_{T\cap F'} \langle Nrd(b_1), \ldots, Nrd(b_n)\rangle_{T\cap F'}.$$

By the Representation Criterion [6, 1.19], $Nrd(b_1) = Nrd(a_1)t_1 + \cdots + Nrd(a_n)t_n$ for some $t_i \in T \cap F'$. Hence, we have $b_1 = Nrd(a_1)(t_1Nrd(b_1)^{-1}b_1) + \cdots + Nrd(a_n)(t_nNrd(b_1^{-1})b_1)$. Since $b_1Nrd(b_1)^{-1} \in T$ and t_i's are in F', each $t_iNrd(b_1)^{-1}b_1 \in T$. This shows that $b_1 \in D_T(\phi)$. □

Proposition 5.6. *Let $\phi = \langle a_1, \ldots, a_n\rangle_T$ be a T-form. The following statements are equivalent:*

(a) *ϕ is isotropic.*
(b) *$\phi \cong_T \langle 1, -1, b_3, \ldots, b_n\rangle_T$ for some $b_3, \ldots, b_n \in S(D^\times,*)$.*
(c) *$\langle Nrd(a_1), \ldots, Nrd(a_n)\rangle_{T\cap F'}$ is an isotropic $T\cap F'$-form.*
(d) *There exists an element $b \in S(D^\times,*)$ such that $\pm b \in D_T(\phi)$.*

Proof. (a) $\Rightarrow$ (b): Suppose $\phi = \langle a_1, \ldots, a_n\rangle_T$ is T-isotropic. Then there exist $t_1, \ldots, t_n \in T$, not all zero, such that $\sum_{i=1}^{n} Nrd(a_i)t_i = 0$. Without loss of generality, we may assume $t_1 \neq 0$. Then

$$-Nrd(a_1)t_1 \in D_T(\langle a_2, \ldots, a_n\rangle_T).$$

By the Representation Criterion, there exist $b_3, \ldots, b_n \in S(D^\times, *)$ such that

$$\langle -Nrd(a_1)t_1, b_3, \ldots, b_n\rangle_T \cong_T \langle a_2, \ldots, a_n\rangle_T.$$

By applying Lemma 5.1(ii) twice, first divided by t_1 and then by $a_1 Nrd(a_1)^{-1}$, we obtain $\langle -a_1, b_3, \ldots, b_n\rangle_T \cong_T \langle a_2, \ldots, a_n\rangle_T$. Hence $\langle a_1, -a_1, b_3, \ldots, b_n\rangle_T \cong_T \langle a_1, \ldots, a_n\rangle_T$. Now (b) follows by using the fact that $\langle a_1, -a_1\rangle \cong_T \langle 1, -1\rangle$.

(b) $\Rightarrow$ (c): By (b) and Lemma 5.1(ii), we obtain

$$\langle Nrd(a_1), \ldots, Nrd(a_n)\rangle_T \cong_T \langle 1, -1, Nrd(b_3), \ldots, Nrd(b_n)\rangle_T.$$

By Lemma 5.2 and [6, Corollary 1.20], $\langle Nrd(a_1), \ldots, Nrd(a_n)\rangle_{T\cap F'}$ is an isotropic $T\cap F'$-form.

(c) $\Rightarrow$ (d): It follows from [6, Corollary 1.20] and the definition of $D_T(\phi)$.

(d) $\Rightarrow$ (a): Observe that by Lemma 5.4, there exist $t_i, t_i' \in T$ such that $b = \sum Nrd(a_i)t_i$ and $-b = \sum Nrd(a_i)t_i'$. Since $b \neq 0$, not all t_i's are zero. Consequently, $0 = \sum Nrd(a_i)(t_i + t_i')$ and not all $t_i + t_i'$ are zero in T. Therefore, ϕ is isotropic. □

Corollary 5.7. *Let T be a weak preordering of $(D, *)$ and $\phi = \langle a_1, \ldots, a_n\rangle_T$:*

$$D_T(\phi)\cap F' = D_{T\cap F'}\langle Nrd(a_1), \ldots, Nrd(a_n)\rangle_{T\cap F'}.$$

Proof. Let $b_1 \in D_T(\phi)\cap F'$. By the Representation Criterion and Lemma 5.2, we see that

$$\langle Nrd(b_1), \ldots, Nrd(b_n)\rangle_{T\cap F'} \cong_{T\cap F'} \langle Nrd(a_1), \ldots, Nrd(a_n)\rangle_{T\cap F'}$$

for suitable $b_2, \ldots, b_n \in S(D^\times, *)$. As $deg\, D$ is odd and $b_1 \in F'$, $Nrd(b_1) = b_1^{2r+1}$ for some integer r. Applying the Representation Criterion for forms over preordering, we conclude $b_1 \in D_{T\cap F'}\langle Nrd(a_1), \ldots, Nrd(a_n)\rangle_{T\cap F'}$. □

We remark here that many other known results in [6] can easily be generalized by applying the above results. We shall not go into the details here. Instead, we go back to the study of Y_D^*. We first prove the converse of Corollary 3.6.

Proposition 5.8. *Suppose $[D:F]$ is odd and T' is an anisotropic $\tilde{\mathscr{J}}(\{1\})\cap F'$-module of F' that contains 1. Then $\tilde{\mathscr{J}}(T')$ is a Baer preordering and $\tilde{\mathscr{J}}(T')\cap F' = T'$. Furthermore, $\tilde{\mathscr{J}}(T') = \{x \in S(D, *): Nrd(x) \in T'\}$ and any normed $\tilde{\mathscr{J}}(\{1\})\cap F'$-semiordering can be extended uniquely to a Baer ordering of $(D, *)$.*

Proof. Clearly, $\tilde{\mathscr{J}}(T')$ satisfies (i), (ii) and (iv) of Definition 3.1. Let $a \in \tilde{\mathscr{J}}(T')$. Suppose there exist f_i's in $\tilde{\mathscr{J}}$ and nonzero a_i's in F' such that $a = \sum_{i=1}^n f_i(a_i)$. Note that $a = \sum_{i=1}^n a_i f_i(1)$. If $a = 0$, then $\langle a_1, \ldots, a_n\rangle_{\tilde{\mathscr{J}}(\{1\})}$ is $\tilde{\mathscr{J}}(\{1\})$-isotropic. By Proposition 5.6, $\langle a_1, \ldots, a_n\rangle_{\tilde{\mathscr{J}}(\{1\})\cap F'}$ is also $\tilde{\mathscr{J}}(\{1\})\cap F'$-isotropic. This is impossible as T' is an anisotropic $\tilde{\mathscr{J}}(\{1\})\cap F'$-module; so $a \neq 0$. This implies $\tilde{\mathscr{J}}(T')\cap -\tilde{\mathscr{J}}(T') = \{0\}$. Hence $\tilde{\mathscr{J}}(T')$ is a Baer preordering. Suppose a defined above is also in F'. Note that

$a \in D_{\tilde{\mathcal{J}}(\{1\})}\langle a_1, \ldots, a_n\rangle_{\tilde{\mathcal{J}}(\{1\})}$. By Corollary 5.7, $a \in D_{\tilde{\mathcal{J}}(\{1\})\cap F'}\langle a_1, \ldots, a_n\rangle_{\tilde{\mathcal{J}}(\{1\})\cap F'}$. As all a_i's are in T' and T' is a $\tilde{\mathcal{J}}(\{1\})\cap F'$-module, $a \in T'$. This proves $\tilde{\mathcal{J}}(T')\cap F' = T'$. Thus by Corollary 3.6, $\tilde{\mathcal{J}}(T') = \{x \in S(D,*)\colon Nrd(x) \in T'\}$. Observe that any normed T'-semiordering Q of F' is an anisotropic $\tilde{\mathcal{J}}(\{1\})\cap F'$-module that contains 1. Therefore, $\{x \in S(D,*)\colon Nrd(x) \in Q\}$ is the unique Baer ordering that extends Q. $\square$

Theorem 5.9. *Let Y_D^* be the space of Baer orderings of $(D,*)$ and $Y/(\tilde{\mathcal{J}}(\{1\})\cap F')$ be the space of normed $\tilde{\mathcal{J}}(\{1\})\cap F'$-semiorderings of F'. Define $\Phi\colon Y_D^* \to Y/(\tilde{\mathcal{J}}(\{1\})\cap F')$ such that $\Phi(P) = P\cap F'$ for all $P \in Y_D^*$. Then Φ is a homeomorphism. In particular, any normed $(\tilde{\mathcal{J}}(\{1\})\cap F')$-semiordering of F' can be extended to a Baer ordering of $(D,*)$.*

Proof. By Corollary 3.6, Φ is well defined. Clearly, its restriction on X_D^* is the mapping ϕ defined in Section 3. As before, Y_D^* and $Y/(\tilde{\mathcal{J}}(\{1\})\cap F')$ can be topologized by using Harrison sets as subbases. Obviously, Φ is continuous. Injectivity follows from Corollary 3.8. Furthermore, when $(D,*)$ is trivial, Φ is clearly a homeomorphism. In case $[D:F]$ is odd, surjectivity follows from Proposition 5.8. Adding to the fact that in Y_D^*, $\{P \in Y_D^*\colon x \in P\} = \{P \in Y_D^*\colon Nrd(x) \in P\}$, we deduce that Φ is an open mapping. Therefore, Φ is a homeomorphism. $\square$

In view of the above theorem, we see that the study of Baer orderings of $(D,*)$ reduces to the study of semiorderings over F' when $[D:F]$ is odd. In order to generalize the above result to arbitrary weak preorderings, we need a new notion of Baer orderings.

Definition 5.10. Let T be a weak preordering. A Baer ordering P is called a *T-Baer ordering* if $P\cap F'$ is a normed T-semiordering of F'.

The notion of T-Baer orderings in analogous to the notion of normal T-semiorderings. Let Y_T be space of T-Baer orderings of $(D,*)$ and $Y/(T\cap F')$ the space of normed $T\cap F'$-semiorderings of F'. Clearly, $\Phi(Y_T) \subset Y/(T\cap F')$. Thus, Φ_T, the restriction of Φ on Y_T, is a mapping from Y_T to $Y/(T\cap F')$.

Theorem 5.11. *Let T be a weak preordering of $(D,*)$. Φ_T defined above is a homeomorphism.*

Proof. As before, we only need to deal with the case when $[D:F]$ is odd. Since we have already proved that Φ is a homeomorphism, we only need to show Φ_T is surjective. For any $Q \in Y/(T\cap F')$, $\Phi^{-1}(Q)\cap F' = Q$. It follows from Corollary 3.8 that $\Phi^{-1}(Q) \in Y_T$ if $Q \in Y/(T\cap F')$. $\square$

To end this paper, we give an application of Theorem 5.11. Let T be a weak preordering. We say T is a SAP preordering if X_T satisfies SAP. In [8], we deal with the special case when $T = \tilde{\mathcal{J}}(\{1\})$.

Theorem 5.12. *Let $(D,*)$ be a $*$-field and T a weak preordering of $(D,*)$. The following conditions are equivalent:*

(a) *X_T satisfies SAP.*

(b) *X_T satisfies WAP.*

(c) *Every three nonarchimedean weak $*$-orderings of X_T can be separated by some $d \in S(D^\times, *)$ from any other nonarchimedean weak $*$-ordering of X_T.*

(d) *Suppose v is a $*$-valuation semicompatible with a T-Baer ordering and Γ_D is its value group. We have either* (i) $|S(\Gamma_D)/\Gamma_T| = 1$, *and $\overline{T \cap F'}$ is a SAP preordering of F', or* (ii) $|S(\Gamma_D)/2\Gamma_T| = 2$, *and $\bar{T}$ is a weak $*$-ordering of $(\bar{D}_v, \bar{*})$.*

(e) *$Y_T = X_T$.*

Proof. When $(D,*)$ is trivial, the notion of weak preorderings and T-Baer orderings reduce, respectively, to preorderings and normed T-semiorderings of the field F'. All the above statements are known to be equivalent. We may therefore assume $[D:F]$ is odd. By Proposition 3.9 and Theorem 5.12, we see that (a), (b), (c) and (e) are, respectively, equivalent to

(a′) $X/(T \cap F')$ is SAP.

(b′) $X/(T \cap F')$ is WAP.

(c′) Every three nonarchimedean orderings of $X/(T \cap F')$ can be separated by some $d \in F'$ from any other nonarchimedean ordering of $X/(T \cap F')$.

(e′) $Y/(T \cap F') = X/(T \cap F')$.

It is well known that (a′), (b′), (c′) and (e′) are all equivalent. We only need to prove (c) $\Rightarrow$ (d) and (d) $\Rightarrow$ (e).

Suppose v is a $*$-valuation of $(D,*)$. We claim that $|S(\Gamma_D):\Gamma_T| = |\Gamma_{F'}:\Gamma_{T \cap F'}|$. By [8, Lemma 1.4], we know that $S(\Gamma_D) = \Gamma_D$ and $|\Gamma_D:\Gamma_{F'}|$ is odd. Therefore, $S(\Gamma_D) = \Gamma_D = 2\Gamma_D + \Gamma_{F'}$. We claim that $\Gamma_T = 2\Gamma_D + \Gamma_{T \cap F'}$. Clearly, we only need to prove $\Gamma_T \subset 2\Gamma_D + \Gamma_{T \cap F'}$. Let $x \in T$. Then $Nrd(x) \in T \cap F'$. As proved in the main theorem of [11, Theorem], $v(x) = v(Nrd(x))/(deg\, D)$. As $deg\, D$ is odd, $v(x) \in v(Nrd(x)) + 2\Gamma_D \subset 2\Gamma_D + \Gamma_{T \cap F'}$. This proves our claim.

(c) $\Rightarrow$ (d): As (c) and (c′) are equivalent, we conclude that $T \cap F'$ is a SAP preordering of F'. Since $v|_{F'}$ is a real valuation, [6, Theorems 17.12, 16.3] imply that $|\Gamma_{F'}:\Gamma_{T \cap F'}| \le 2$. Moreover, when $|\Gamma_{F'}:\Gamma_{T \cap F'}| = 1$, $\overline{T \cap F'}$ is a SAP preordering of F'. Whereas if $|\Gamma_{F'}:\Gamma_{T \cap F'}| = 2$, $\overline{T \cap F'}$ is an ordering of $\overline{F'}$. By the above claim, we see that $|S(\Gamma_D):\Gamma_T| = |\Gamma_{F'}:\Gamma_{T \cap F'}| \le 2$. We only have to deal with case when $|S(\Gamma_D):\Gamma_T| = |\Gamma_{F'}:\Gamma_{T \cap F'}| = 2$. Now, $\overline{T \cap F'}$ is an ordering of $\overline{F'}$. But as proved in [8, Theorem 3.1], $\overline{T \cap F'}$ extends uniquely to a $*$-ordering of $(Z(\bar{D}_v), \bar{*})$, $\bar{T} \cap Z(\bar{D}_v)$ must be a weak $*$-ordering of $(Z(\bar{D}_v), \bar{*})$. Since $\bar{T}$ is a weak preordering, it follows from Corollary 3.8 that $\bar{T}$ is a weak $*$-ordering.

(d) $\Rightarrow$ (e): Let P be a T-Baer ordering. It suffices to show that $P \cap F'$ is an ordering of F'. By assumption, $|\Gamma_{F'}:\Gamma_{T \cap F'}| = |S(\Gamma_D):\Gamma_T| \le 2$.

Case (i): $|\Gamma_{F'}:\Gamma_{T \cap F'}| = 1$. Since $\overline{P \cap F'}$ is a $\overline{T \cap F'}$-semiordering of $\overline{F'}$ and $\overline{T \cap F'}$ is SAP, $\overline{P \cap F'}$ is an ordering. Applying [6, Remark 15.9], we see that $P \cap F'$ is an ordering.

Case (ii): $|\Gamma_{F'} : \Gamma_{T\cap F'}| = 2$. We can assume v is compatible with P. (Otherwise, we can apply [8, Corollary 4.11] to replace v by a coarsening of v.) Now, $\bar{T}$ is an ordering of $(\bar{D}_v, \bar{*})$. Hence, $\overline{T\cap F'}$ is an ordering of $\bar{F}$. Clearly, $P\cap F'$ is an ordering of F'. $\square$

Remark. As we have observed earlier, when $T = \bar{\mathscr{J}}(\{1\})$, Y_T is just the space of all Baer orderings. Therefore, Theorem 5.12 generalizes [8, Theorem 5.6].

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